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Estimation of Boundary Conditions in Nonlinear Inverse Heat Conduction Problems

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A sequential method is proposed to estimate boundary conditions for nonlinear heat conduction problems. An inverse solution is deduced from a finite element method, the concept of the future time, and a modified Newton–Raphson method. The undetermined boundary condition at each time step is denoted as the unknown variable in a set of nonlinear equations, which are formulated from the measured temperature and the calculated temperature. Then, an iterative process is used to solve the set of nonlinear equations. No selected function is needed to represent the undetermined function in advance. Two examples are used to demonstrate the characteristics of the proposed method. The close agreement between the exact values and the estimated results confirms the validity and accuracy of the proposed method. The results show that the proposed method is an accurate, stable, and efficient method to determine the boundary conditions in two-dimensional nonlinear inverse heat conduction problems.

Nomenclature

$C(T)$	=	temperature-dependent specific heat capacity
h	=	convection heat transfer coefficient
$\boldsymbol{J}$	=	error function
$k_x(T)$	=	temperature-dependent thermal conductivity along the x axis
$k_y(T)$	=	temperature-dependent thermal conductivity along the y axis
N_t	=	number of the temporal measurements
q_n	=	heat flux
q_0	=	value of heat flux on Γ_q
r	=	number of the future time
T	=	temperature
T_b	=	temperature on Γ_T
T_f	=	temperature at which convection occurs
T_m	=	temperature at the mth time step
T_∞	=	bulk temperature
t	=	temporal coordinate
V	=	general spatial domain
X_m	=	sensitivity function of T with respect to the undetermined condition at m time step
x, y	=	spatial coordinate
$\boldsymbol{x}_0$	=	vector of the initial guess
Y	=	measured temperature
Γ_c	=	boundary on which convection condition is prescribed
Γ_q	=	boundary on which heat flux condition is prescribed
Γ_T	=	boundary on which temperature condition is prescribed
$\boldsymbol{\Delta}$	=	increment of the search step
ε, δ	=	value of the stopping criterion
λ	=	random number
ρ	=	density
σ	=	standard deviation of measurement error
$\boldsymbol{\Phi}$	=	calculated temperature minus measured temperature
$\boldsymbol{\Phi}$	=	vector constructed from $\boldsymbol{\Phi}$
$\boldsymbol{\Phi}_c$	=	calculated temperature
$\boldsymbol{\Phi}_{\text{meas}}$	=	measured temperature
$\hat{\boldsymbol{\Phi}}_u$	=	component of vector $\boldsymbol{\Phi}$
ϕ_m^{c,i_c}	=	unknown convection condition at i_cth grid and mth time step
ϕ_m^{q,i_q}	=	unknown heat flux condition at i_qth grid and mth time step
ϕ_m^{T,i_T}	=	unknown temperature condition at i_Tth grid and mth time step
$\boldsymbol{\Psi}$	=	sensitivity matrix
$\boldsymbol{\Psi}_{u,v}$	=	component of vector $\boldsymbol{\Psi}$

Subscripts

i, j, m, u, v	=	indices
n_c	=	number of unknown grids of convection boundary
n_p	=	number of grids at spatial coordinate
n_q	=	number of unknown grids of heat flux boundary
n_T	=	number of unknown grids of temperature boundary
p	=	number of spatial measurement

Superscripts

exact	=	exact temperature
i_c	=	grid number of the estimated convection temperature
i_q	=	grid number of the estimated flux function
i_T	=	grid number of the estimated temperature function
meas	=	measured temperature

Introduction

THE determination of boundary conditions from measured temperature profiles is an inverse boundary estimation problem. Boundary estimation problems have been widely applied in many design and manufacturing problems especially when the direct measurements of the surface conditions are not possible such as the measurement of the heat flux or temperature at the inner surface of a heated pipe, at the inside of a combustion chamber, at the outer surface of a reentry vehicle, or at the tool–work interface of a machine cutting. Through the inverse technique, the unknown boundary conditions can be deduced indirectly from the temperature measurements at different locations within the medium. In past research, most of the inverse techniques are confined to problems with temperature-independent thermal properties, which is the linear heat conduction equation. However, for most practical engineering problems, the thermal properties are temperature dependent and lead the heat conduction equation to a nonlinear form. The determination of a thermal system with temperature-dependent properties

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is more difficult than that with temperature-independent properties because the magnitude of the thermal properties has a significant influence on the analysis of temperature distribution, heat flow rate, and thermal instability problems.

The inverse solution is often unstable even if the measured data have a slight variation in the experimental measurements. Two important methods have been used to improve the stability of the estimation. One is the regularization method,[1] and the other is the concept of the future time.[2] The regularization function penalizes large variations in the unknown quantities by adding a term to the nonlinear least-squares-error function. The concept of future time makes assumptions about the behavior of the experimental data at future time steps, which is added to the measurement in the nonlinear least-squares formulation. Analysis and application of the regularization method and of the concept of future time have been discussed.[3–9] Other inverse methods include the iterative regularization method,[5] the sequential regularization method,[3] dynamic programming,[10] mollification method,[11] adjoint equation approach coupled to the conjugate-gradient method,[12] genetic algorithms,[13] symbolic sequential method,[14] and numerical sequential method.[15] However, few works have been done to estimate the boundary condition in the nonlinear inverse problem.[16,17]

An accurate estimation of a nonlinear inverse problem is more difficult than that of a linear inverse problem because the magnitude of the thermal properties has a significant influence on the analysis of thermal behavior. For example, Dorai and Tortorelli[17] use Newton's method to estimate the boundary conditions in linear and nonlinear inverse problems. In the linear inverse problem, the solution can be achieved in certain iterations (case 3, Ref. 17). In the nonlinear inverse problem, the inverse solution does not converge after 10 iterations when Newton' method is adopted, and the solution converges at 11 iterations when the combined Broyton–Fletcher–Goldfarb–Shanno– (BFGS–) Newton method is used. However, the solution still failed to converge at 20 time steps for case 4 (Ref. 17) when the combined BFGS–Newton method is used. Therefore, it is necessary to develop an inverse algorithm to estimate the boundary condition for the nonlinear inverse heat conduction problem.

In this paper, a sequential method combined with the concept of future time is proposed to solve the problems step by step. Also a modified Newton–Raphson method[18] is used to search the inverse solution at each time step. In the proposed approach, the determination of the boundary conditions at each time step includes two phases: the process of direct analysis and the process of inverse analysis. In the direct analysis process, the boundary conditions are assumed as known values and then the temperature field of the heat conduction equation is solved through a finite element method. Solutions from this process are substituted into the sensitivity analysis and integrated with the available temperature measured at the sensors' locations. Thus, a set of nonlinear equations is formulated for the process of the inverse estimation. In the inverse analysis process, an iterative method is used to guide the exploring points systematically to approach the undetermined boundary conditions. Then, the intermediate boundaries are substituted for the unknown properties in the analysis that follows. As such, several iterations are needed to obtain the undetermined boundary conditions. In the present research, the proposed method formulates the problem from the difference between the calculated temperature and the one measured directly. Therefore, the inverse formulation derived from the proposed method is simpler than that from the nonlinear least-squares method.

The paper includes the following sections. First, current research in estimating the boundary conditions is introduced, and the feature of using the proposed method in the problem is also stated. In the next section, the characteristics of solving the inverse problem are delineated, and the content of the concept of future time, the direct problem, the sensitivity problem, and the algorithm of the proposed method are presented. The criterion to stop the iterative process is illustrated. Then in the next section, the computational algorithm of the proposed method is shown. Two examples are employed to demonstrate and discuss the results of the proposed method in the following section. In the final section, the overall contribution of this research to the field of inverse heat conduction problem is discussed.

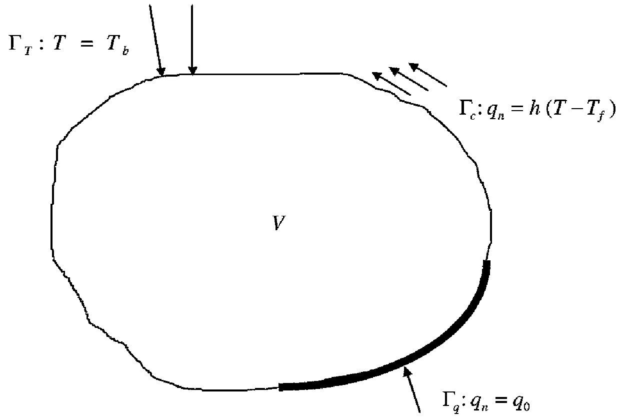

Fig. 1 Two-dimensional body with various types of boundary conditions.

Problem Statement

The inverse problem is to find one part of the boundary condition in a two-dimensional body when the temperature measurements at the other part are given. Consider a two-dimensional body that is subjected to the following three boundary condition types (Fig. 1): 1) the specified temperature $T = T_b$ on Γ_T, 2) the specified heat flux $q_n = q_0$ on Γ_q, and 3) the specified convection $q_n = h(T - T_f)$ on Γ_c. The interior of the body is V, and the boundary is $\Gamma = \Gamma_T \cup \Gamma_q \cup \Gamma_c$. Furthermore, q_n is the heat flux normal to the boundary. The sign convention adopted here for specifying q_0 is that $q_0 > 0$ if heat is flowing out of the body, whereas $q_0 < 0$ if heat is flowing into the body. The transient heat conduction problem is governed by the following equations:

$$\frac{\partial}{\partial x}\left(k_x(T)\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y(T)\frac{\partial T}{\partial y}\right) = \rho C(T)\frac{\partial T}{\partial t} \qquad (x_i, y_i) \in V \quad (1)$$

$$T(x_i, y_i, t) = T_b(x_i, y_i, t), \qquad (x_i, y_i) \in \Gamma_T \quad (2)$$

$$q_n(x_i, y_i, t) = q_0(x_i, y_i, t), \qquad (x_i, y_i) \in \Gamma_q \quad (3)$$

$$q_n(x_i, y_i, t) = h[T(x_i, y_i, t) - T_f(t)], \qquad (x_i, y_i) \in \Gamma_c \quad (4)$$

$$T(x_i, y_i, 0) = T_0(x_i, y_i), \qquad (x_i, y_i) \in \Gamma \cup V \quad (5)$$

where T is the temperature field $T(x_i, y_i, t)$. Here $k_x(T)$ and $k_y(T)$ are the thermal conductivity, and $\rho C(T)$ is the heat capacity per unit volume.

The inverse problem is to estimate the boundary conditions of the nonlinear heat conduction problem when the temperature field is measured at the known boundary. The thermal properties of the problem are temperature dependent and lead to the thermal instability problem.

Hsu et al.[19] mentioned that it is difficult to have an algorithm that can estimate the heat flux and the surface temperature through the same technique in the multidimensional inverse heat conduction problem. However, the proposed method is able to estimate the different types of boundary conditions in the same algorithm.

Proposed Sequential Method to Estimate Boundary Conditions

In each time step, an iterative algorithm is used to estimate simultaneously part of the boundary conditions in a two-dimensional body, whereas the temperature measurements at the other part are given in a transient heat conduction experiment. Some procedures are needed to solve the inverse problem. They are the direct problem, the sensitivity problem, the operational algorithm, and the stopping criterion. The direct problem is used to determine the temperature distribution, and the sensitivity problem is used to find the search step in the inverse problem. The operational algorithm is used to fulfill the process of the inverse analysis when the solutions of the direct problem and the sensitivity problem are available. Finally, the stopping criterion is shown to stop the iterative process.

Direct Problem

The proposed method is based on a sequential algorithm, and the inverse solution is solved at each time step. Therefore, Eqs. (1–5) are limited to only one time step, and the transient heat conduction problem at $t = t_m$ is governed by the following equations:

$$\frac{\partial}{\partial x}\left(k_x(T_m)\frac{\partial T_m}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y(T_m)\frac{\partial T_m}{\partial y}\right) = \rho C(T_m)\frac{\partial T_m}{\partial t}$$

$$(x_i, y_i) \in V \quad \text{at} \quad t = t_m \qquad (6)$$

$$T(x_i, y_i, t_m) = \phi_m^{T,i_T}, \qquad (x_i, y_i) \in \Gamma_T \qquad (7)$$

$$q_n(x_i, y_i, t_m) = \phi_m^{q,i_q}, \qquad (x_i, y_i) \in \Gamma_q \qquad (8)$$

$$q_n(x_i, y_i, t_m) = h\left[T(x_i, y_i, t_m) - \phi_m^{c,i_c}\right], \qquad (x_i, y_i) \in \Gamma_c \qquad (9)$$

$$T(x_i, y_i, t_{m-1}) = T_{m-1}, \qquad (x_i, y_i) \in \Gamma \cup V \qquad (10)$$

where

$$T_m = T(x_i, y_i, t_m), \qquad \phi_m^{T,i_T} = T_b(x_i, y_i, t_m)$$

$$\phi_m^{q,i_q} = q_0(x_i, y_i, t_m), \qquad \phi_m^{c,i_c} = T_f(t_m)$$

Here, ϕ_m^{T,i_T}, ϕ_m^{q,i_q}, and ϕ_m^{c,i_c} are the unknown boundaries of the temperature condition, the heat flux condition, and the convection condition, respectively, and i_T, i_q, and i_c are the indices of the sensors' locations.

The inverse solution of the problem, especially the nonlinear problem, is ill posed, and it is often unstable when the measured data have a slight variation in the experimental measurements. The concept of future time is used to improve the stability of the estimation in this research. The concept of future time makes assumptions about the behavior of the experimental data at future time steps, which are included in the measurement to estimate the present state. In the present research, the proposed method formulates the problem from the difference between the calculated temperature and the one measured directly. The equation solver, rather than the optimization algorithm, solves the inverse problem.

When $t = t_m$, the estimated condition between $t = t_1$ and $t = t_{m-1}$ has been evaluated, and the problem is to estimate the strength of the boundary conditions at $t = t_m$. To stabilize the estimated results in the inverse algorithms, the sequential procedure is to assume temporally that several future values of the estimation are constant.[2] Then, the unknown conditions at the future time are assumed to be equal, that is,

$$\phi_{m+1}^{T,i_T} = \phi_{m+2}^{T,i_T} = \cdots = \phi_{m+r-2}^{T,i_T} = \phi_{m+r-1}^{T,i_T} = \phi_m^{T,i_T}$$

$$\phi_{m+1}^{q,i_q} = \phi_{m+2}^{q,i_q} = \cdots = \phi_{m+r-2}^{q,i_q} = \phi_{m+r-1}^{q,i_q} = \phi_m^{q,i_q}$$

$$\phi_{m+1}^{c,i_c} = \phi_{m+2}^{c,i_c} = \cdots = \phi_{m+r-2}^{c,i_c} = \phi_{m+r-1}^{c,i_c} = \phi_m^{c,i_c} \qquad (11)$$

Here r is the future time.

The direct problem equations (6–10) are solved in r steps (from $t = t_m$ to t_{m+r-1}) and the undetermined boundaries are set by Eq. (11).

The direct problem is used to generate the simulated temperature when the values of undetermined boundaries are specified. It is a nonlinear problem because the coefficients in Eq. (6) are functions of temperature. Therefore, a finite element method is used to solve the direct problem iteratively when the initial conditions are given. Then, the results from the direct analysis can be substituted into the sensitivity equation and lead to a sensitivity analysis.

Sensitivity Problem

In the proposed method, the modified Newton–Raphson method is adopted to solve the inverse problem in that the sensitivity analysis is necessary to decide the search step in each iteration. The derivative $\partial/\partial\phi_m^{*,i*}$ is taken at both sides of Eqs. (6–10). Then, we have

$$\frac{\partial}{\partial x}\left(k_x(T_m)\frac{\partial X_m}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y(T_m)\frac{\partial X_m}{\partial y}\right) = \rho C(T_m)\frac{\partial X_m}{\partial t}$$

$$(x_i, y_i) \in V, \qquad t = t_m \qquad (12)$$

$$X(x_i, y_i, t_m) = \frac{\partial \phi_m^{T,i_T}}{\partial \phi_m^{*,i*}}, \qquad (x_i, y_i) \in \Gamma_T \qquad (13)$$

$$\frac{\partial q_n(x_i, y_i, t_m)}{\partial \phi_m^{*,i*}} = \frac{\partial \phi_m^{q,i_q}}{\partial \phi_m^{*,i*}}, \qquad (x_i, y_i) \in \Gamma_q \qquad (14)$$

$$\frac{\partial q_n(x_i, y_i, t_m)}{\partial \phi_m^{*,i*}} = \frac{-\partial h \phi_m^{c,i_c}}{\partial \phi_m^{*,i*}}, \qquad (x_i, y_i) \in \Gamma_c \qquad (15)$$

$$X(x_i, y_i, m-1) = X_{m-1} = 0, \qquad (x_i, y_i) \in \Gamma \cup V \qquad (16)$$

where

$$X_m = \frac{\partial T(x_i, y_i, t_m)}{\partial \phi_m^{*,i*}}$$

Equations (12–16) are the mathematical equations for sensitivity coefficient X_m that can be explicitly found if $\phi_m^{*,i*}$ and T_m are known. The equations are linear, and the dependent variable X_m is with respect to the independent variables x, y, and t. Therefore, the sensitive data can be determined directly through a finite element method.

Modified Newton–Raphson Method

The Newton–Raphson method[20] has been widely adopted to solve a set of nonlinear equations. This method is applicable to solve the nonlinear problem when the number of the equations and the number of the unknown variables are the same. In the inverse problem, the number of equations is usually larger than the number of variables; therefore, a modified version of the Newton–Raphson method is necessary to deal with the inverse problem.

In the present research, the proposed method formulates the problem from the comparison between the calculated temperature and the one measured directly. Therefore, the calculated temperature $\mathbf{\Phi}_c(\bar{i}, j)$ and the measured temperature $\mathbf{\Phi}_{\text{meas}}(\bar{i}, j)$ at the $\bar{i}$ grid of the spatial coordinate and at the j grid of the temporal coordinate first need to be evaluated. Then, the estimation of the unknown boundary at each time step and can be recast as the solution of a set of nonlinear equations:

$$\mathbf{\Phi}(\bar{i}, j) = \mathbf{\Phi}_c(\bar{i}, j) - \mathbf{\Phi}_{\text{meas}}(\bar{i}, j) = 0 \qquad (17)$$

where $\bar{i} = 1, 2, \ldots, p$ and $j = m, m+1, \ldots, m+r-1$ and where r is the future time number.

This set of equations has $n_T + n_q + n_c$ variables, that is, the number of undetermined boundaries. The number of equations is the number of the measured point p times the future time number r. If the number of independent equations is greater than the number of the variables, that is, $p \times r > n_T + n_q + n_c$, then the set of equation can be solved through the modified method.

This procedure is detailed as follows:

Substitute the index j from m to $m+r-1$ and the index $\bar{i}$ from 1 to p into Eq. (17). We then have

$$\boldsymbol{\Phi} = [\boldsymbol{\Phi}(\bar{i}, m), \boldsymbol{\Phi}(\bar{i}, m+1), \boldsymbol{\Phi}(\bar{i}, m+2), \ldots, \boldsymbol{\Phi}(\bar{i}, m+r-1)]^T$$
$$= [\boldsymbol{\Phi}(1, m), \boldsymbol{\Phi}(1, m+1), \boldsymbol{\Phi}(1, m+2), \ldots, \boldsymbol{\Phi}(1, m+r-1)|$$
$$\boldsymbol{\Phi}(2, m), \boldsymbol{\Phi}(2, m+1), \boldsymbol{\Phi}(2, m+2), \ldots, \boldsymbol{\Phi}(2, m+r-1)|$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\boldsymbol{\Phi}(p, m), \boldsymbol{\Phi}(p, m+1), \boldsymbol{\Phi}(p, m+2), \ldots, \boldsymbol{\Phi}(p, m+r-1)]^T$$
$$= \{\hat{\boldsymbol{\Phi}}_u\} \tag{18}$$

where $\hat{\boldsymbol{\Phi}}_u$ is the component of vector $\boldsymbol{\Phi}$.

The undetermined coefficients are set as follows:

$$\boldsymbol{x} = \left[\phi_m^{T,1}, \phi_m^{T,2}, \ldots, \phi_m^{T,n_T} \middle| \phi_m^{q,1}, \phi_m^{q,2}, \ldots, \phi_m^{q,n_q}\right.$$
$$\left|\phi_m^{c,1}, \phi_m^{c,2}, \ldots, \phi_m^{c,n_c}\right]^T = \lfloor x_1, x_2, x_3, x_4, \ldots, x_{n_T+n_q+n_c}\rfloor^T$$
$$= \{x_v\} \tag{19}$$

where x_v is the component of vector $\boldsymbol{x}$.

The derivative of $\hat{\boldsymbol{\Phi}}_u$ with respect to x_v is solved through Eqs. (12–16) and can be expressed as follows:

$$\boldsymbol{\Psi}_{u,v} = \frac{\partial \hat{\boldsymbol{\Phi}}_u}{\partial x_v} \tag{20}$$

The sensitivity matrix $\boldsymbol{\Psi}$ can be defined as follows:

$$\boldsymbol{\Psi} = \{\boldsymbol{\Psi}_{u,v}\} \tag{21}$$

where $u = 1, 2, 3, \ldots p \times r$ and $v = 1, 2, 3, \ldots, n_T + n_q + n_c$ and where $\boldsymbol{\Psi}_{u,v}$ is the element of $\boldsymbol{\Psi}$ at the uth row and the vth column.

The starting vector $\boldsymbol{x}_0$ can be shown as

$$\boldsymbol{x}_0 = \lfloor x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, \ldots, x_{n_T+n_q+n_c}\rfloor^T \tag{22}$$

With the derivations from Eqs. (18–22), we have the following equation:

$$\boldsymbol{x}_{\lambda+1} = \boldsymbol{x}_\lambda + \boldsymbol{\Delta}_\lambda \tag{23}$$

where $\boldsymbol{\Delta}_\lambda$ is a linear least-squares solution for a set of overdetermined linear equations, which can be derived as follows:

$$\boldsymbol{\Psi}(\boldsymbol{x}_\lambda)\boldsymbol{\Delta}_\lambda = -\boldsymbol{\Phi}(\boldsymbol{x}_\lambda) \tag{24}$$

$$\boldsymbol{\Delta}_\lambda = -[\boldsymbol{\Psi}^T(\boldsymbol{x}_\lambda)\boldsymbol{\Psi}(\boldsymbol{x}_\lambda)]^{-1}\boldsymbol{\Psi}^T(\boldsymbol{x}_\lambda)\boldsymbol{\Phi}(\boldsymbol{x}_\lambda) \tag{25}$$

The preceding derivation is applied at each time step. This method is implemented in the multisensors' measurement. Under this condition, the number of the elements in Eq. (18) is based on the number of measured locations and the number of future times.

Stopping Criteria

The modified Newton–Raphson method [Eqs. (23–25)] is used to determine the unknown vector $\boldsymbol{x}$ defined by Eq. (19). The step size $\boldsymbol{\Delta}_\lambda$ goes from $\boldsymbol{x}_\lambda$ to $\boldsymbol{x}_{\lambda+1}$, and it is determined from Eq. (23). Once $\boldsymbol{\Delta}_\lambda$ is calculated, the iterative process to determine $\boldsymbol{x}_{\lambda+1}$ is executed until the stopping criterion are satisfied.

The discrepancy principle[1,4] is widely used to evaluate the value of the stopping criterion in the inverse technique. However, the stopping criterion generated from the discrepancy principle does not guarantee the convergence of the inverse solution. Therefore, two criteria used by Frank and Wolfe[21] are chosen to assure the convergence and to stop the iteration:

$$\|\boldsymbol{x}_{\lambda+1} - \boldsymbol{x}_\lambda\| \le \delta\|\boldsymbol{x}_{\lambda+1}\| \tag{26}$$

$$\|\boldsymbol{J}(\boldsymbol{x}_{\lambda+1}) - \boldsymbol{J}(\boldsymbol{x}_\lambda)\| \le \varepsilon\|\boldsymbol{J}(\boldsymbol{x}_{\lambda+1})\| \tag{27}$$

where

$$\|\boldsymbol{J}(\boldsymbol{x}_{\lambda+1})\| = \sum_{i=1}^{p}\sum_{j=1}^{r}[\boldsymbol{\Phi}_c(\bar{i}, j) - \Phi_m(\bar{i}, j)]^2 \tag{28}$$

where δ and ε are small positive values. The values of δ and ε are the converge tolerances.

Computational Algorithm

The procedure for the proposed method can be summarized as follows. First, we choose the future time number r; the mesh configuration of the problem domain; the temporal size $\boldsymbol{\Delta}t$; the measured grids, that is, $i_1, i_2, \ldots, i_p$; and the estimated grids, that is, $i_T^1, i_T^2, \ldots, i_T^{n_T}, i_q^1, i_q^2, \ldots, i_q^{n_q}$, and $i_c^1, i_c^2, \ldots, i_c^{n_c}$. Given are overall convergence tolerance δ and ε and the initial guess $\boldsymbol{x}_0$. The value of $\boldsymbol{x}_\lambda$ is known at the λth iteration. Then, the iterative procedure can be summarized as follows:

1) Let $j = m$, and the temperature distribution at $\{T_{j-1}\}$ is known.
2) Collect the measurements $\boldsymbol{\Phi}_{\text{meas}}(\bar{i}, j)$, which are $Y_j^{i_1}, Y_{j+1}^{i_1}, \ldots, Y_{j+r-1}^{i_1}, Y_j^{i_2}, Y_{j+1}^{i_2}, \ldots, Y_{j+r-1}^{i_2}, \ldots Y_j^{i_p}, Y_{j+1}^{i_p}, \ldots, Y_{j+r-1}^{i_p}$.
3) Assume the initial guess $\boldsymbol{x}_0$.
4) Solve the direct problem [Eqs. (6–10)], and compute the calculated temperature $\boldsymbol{\Phi}_c(\bar{i}, j)$.
5) Integrate the calculated temperature $\boldsymbol{\Phi}_c(\bar{i}, j)$ with the measured temperature $\boldsymbol{\Phi}_{\text{meas}}(\bar{i}, j)$ to construct $\boldsymbol{\Phi}$.
6) Calculate the sensitivity matrix $\boldsymbol{\Psi}$ through Eqs. (12–16).
7) With $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ known, compute the step size $\boldsymbol{\Delta}_\lambda$ from Eq. (25).
8) With $\boldsymbol{\Delta}_\lambda$ and $\boldsymbol{x}_\lambda$ known, compute $\boldsymbol{x}_{\lambda+1}$ from Eq. (23).
9) Terminate the process if the stopping criterion [Eqs. (26) and (27)] is satisfied. Otherwise return to step 4.
10) Terminate the process if the final time step is attached. Otherwise, let $j = m + 1$ return to step 2.

Results and Discussion

In this section, problems defined from Eqs. (1–5) are used as examples to estimate the unknown boundary conditions. Two examples are used to demonstrate that the proposed method can estimate the boundary conditions accurately. In the first example, a finned heat sink with temperature-dependent thermal conductivity is cooled by free convection to air. The measured temperature in example 1 is calculated from Eqs. (1–5) when the heat flux entered at the base of the sink is preselected. Two kinds of boundaries, that is, the quantity of the heat flux entering the sink and the temperature at the base of sink, are estimated from the temperature measured at the outer surface of the heat sink. In the second example, a rectangular slab with temperature-dependent thermal conductivity and temperature-dependent thermal heat capacity is insulated at all sides except the topside. Three presumed heat fluxes are applied at the top surface of the slab, and they are estimated from the measured temperature at the bottom surface of the slab. The simulated temperature is generated from the exact temperature in each problem, and it is presumed to have measurement errors. In other words, the random errors of measurement are added to the exact temperature. It can be shown in the following equation:

$$T_{i,j}^{\text{meas}} = T_{i,j}^{\text{exact}} + \lambda_{i,j}\sigma \tag{29}$$

where the subscripts i and j are the grid number of spatial coordinate and temporal coordinate, respectively. $T_{i,j}^{\text{exact}}$ in Eq. (29) is the exact temperature, $T_{i,j}^{\text{meas}}$ is the measured temperature, σ is the standard deviation of measurement errors, and $\lambda_{i,j}$ is a random number. The value of $\lambda_{i,j}$ is calculated by the IMSL subroutine DRNNOR[22] and chosen over the range $-2.576 < \lambda_{i,j} < 2.576$, which represents the 99% confidence bound for the measured temperature.

For example 1, consider a finned heat sink with the cross section as shown in Fig. 2. The finned surface is cooled by free convection to air at 25°C. The value of the thermal conductivity is $k = 386 + 379T + 369T^2$ W/m°C, and the value of the heat capacity per unit volume is $\rho C = 8954 \times 383$ J/m^3°C. The film coefficients of the fin are $h = 1.42[(T - T_\infty)/0.8268]^{0.25}$ and $h = 1.32[(T - T_\infty)/0.2362]^{0.25}$ W/m^2°C on the vertical faces and

Table 1 Relative average errors of example 1

Boundary type	μ
Heat flux $q_0(t)$	
$\sigma = 0.5, r = 2$	0.027871
$\sigma = 0.5, r = 4$	0.011441
$\sigma = 1, r = 4$	0.021828
Temperature $T_b(t)$	
$\sigma = 1, r = 1$	0.006204
$\sigma = 2, r = 1$	0.01241
$\sigma = 3, r = 1$	0.018619
$\sigma = 4, r = 1$	0.24825
$\sigma = 5, r = 1$	0.031033

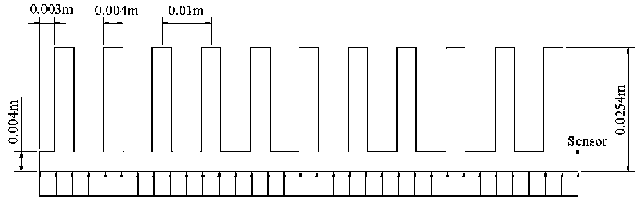

Fig. 2 Boundary condition, location of the sensor, and dimensions of the finned heat sink in example 1.

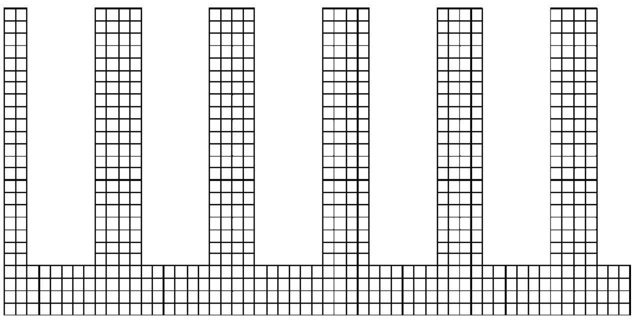

Fig. 3 Mesh configuration with 868 nodes and 682 elements for a finned heat sink.

the horizontal faces, respectively. It is initially at a uniform temperature $T_0 = 25°C$, and then suddenly a uniform heat flux of 7718 W/m^2 enters at the base.

The temporal domain is from 36 to 3600 s with 36-s increments for the example. One thermocouple is located at the right corner of the sink (Fig. 2). A four-node quadratic element is used to generate the mesh of the model and leads to a finite element model with 868 nodes and 682 elements (Fig. 3). The boundary, along with the vertical symmetric line, is insulated. Two types of the boundary conditions are discussed in this example. One is the estimation of the heat flux $q_0(t)$ that enters the base of sink, and the other is the estimation of the temperature $T_b(t)$ that is at the base of the sink.

To investigate the deviation of the estimated results from the error-free solution, the relative average errors for the estimated solutions are defined as follows:

$$\mu = \frac{1}{N_t} \sum_{j=1}^{N_t} \left| \frac{f - \hat{f}}{\hat{f}} \right| \tag{30}$$

where f is the estimated result with measurement errors and $\hat{f}$ is the estimated result without measurement errors. N_t is the number of the temporal steps. It is clear that a smaller value of μ indicates a better estimation and vice versa.

When measurement errors are considered, the relative average errors of the estimated results are shown in Table 1. From the results, it is seen that the heat flux estimation is less accurate than that of the temperature estimation in the problem. In other words, the heat flux estimation is more sensitive to the measurement error than that of the temperature estimation. For example, when $\sigma = 1$, the value of μ is 0.021828 in the heat flux estimation but 0.006204 in the temperature condition estimation. It is also interesting to investigate the relationship among the average relative error μ, the standard deviation of measurement errors σ, and the future time number r. In the flux condition estimation, when $\sigma = 0.5$, a better estimation appears if the future time number increases.

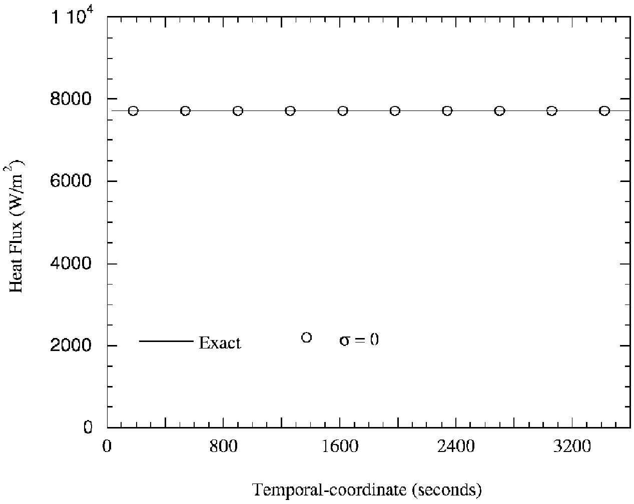

Fig. 4 Estimation of heat flux in example 1 when $\sigma = 0$ and $r = 1$.

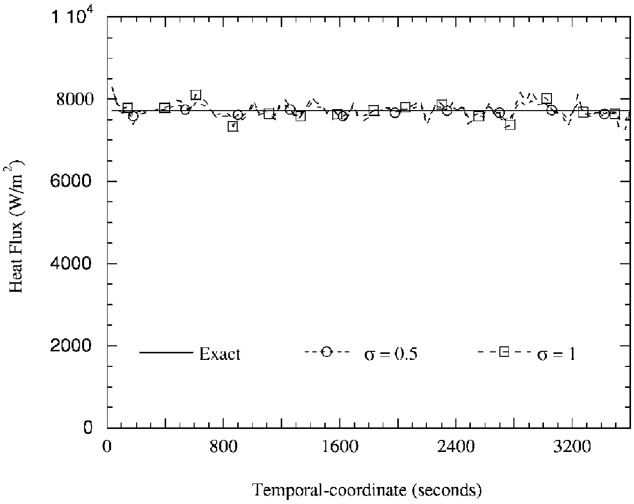

Fig. 5 Estimation of heat flux in example 1 when $\sigma = 0.5$ and $r = 4$ and $\sigma = 1$ and $r = 4$.

When $\sigma = 0$ and $r = 1$, the estimated results are an excellent approximation of the exact solution in the heat flux estimation (Fig. 4). The estimated results with measurement error are shown in Fig. 5. The estimated temperature approaches to a steady state when the time increases (Fig. 6). In general, large errors make the estimated results diverge from the error-free solution. For example, the results shown in Fig. 6 have the maximum deviation from their exact solutions when $\sigma = 5$. The average number of iterations in the heat flux estimation is about 11 at each time step. (For example, the total number iterations is 1164 when $\sigma = 0.5$ and $r = 4$; it is 1147 when $\sigma = 1$ and $r = 4$.) The average number of iterations in the temperature estimation is under seven at each time step. (For example, the total number iterations is 665 when $\sigma = 0$ and $r = 1$; it is 630 when $\sigma = 5$ and $r = 1$.) From the results, the heat flux estimation is shown to need a higher number of iterations to approach a solution than the temperature estimation. In other words, the convergence of the temperature estimation is better than that of the heat flux estimation.

For example 2, consider a rectangular slab in which all sides are insulated except the topside. The adiabatic conditions are applied to the left, right, and bottom surfaces. The slab is initially at a uniform temperature $T_0 = 0°C$, and then suddenly three boundary conditions are applied to the top surface of the slab. The values of the orthotropic

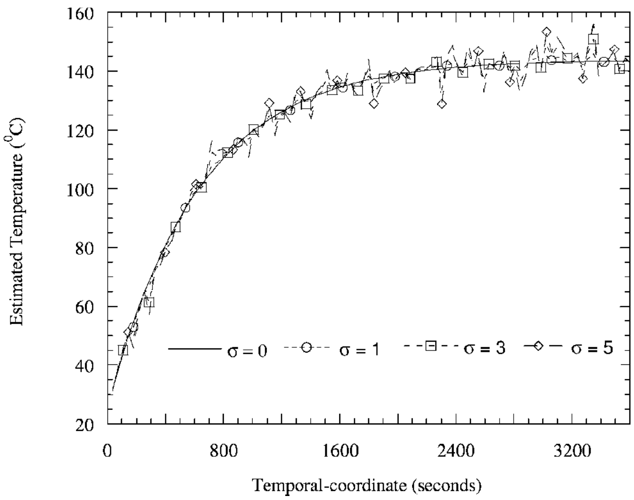

Fig. 6 Estimation of temperature at the base of the sink in example 1 when $r = 1$ and $\sigma = 0, 1, 3$, and 5.

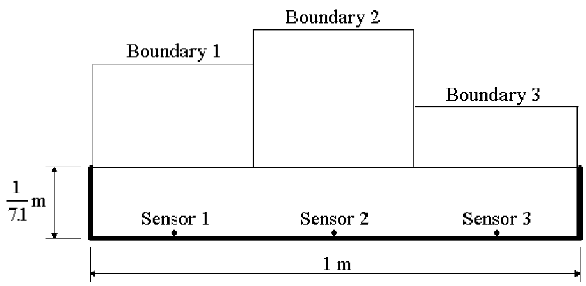

Fig. 7 Boundary conditions, locations of sensors, and dimensions of slab in example 2.

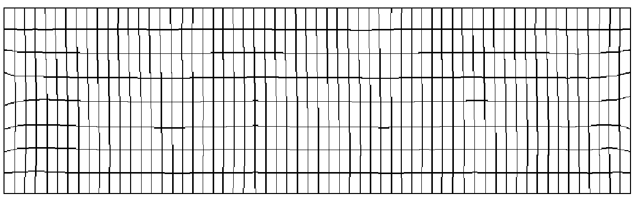

Fig. 8 Mesh configuration with 1577 nodes and 480 elements for the rectangular slab in example 2.

thermal properties are

$$k_x = 0.6081 + 0.00125T + 0.0005T^2 \text{ W/m} \cdot {}^\circ\text{C}$$

$$k_y = 0.7 + 0.00725T + 0.0004T^2 \text{ W/m} \cdot {}^\circ\text{C}$$

$$\rho C = 0.9 + 0.001T \text{ J/m}^3 \cdot {}^\circ\text{C}$$

The temporal domain is from 0.02 to 2 s with 0.02-s increments for the example. Three thermocouples are located at the bottom surface of the slab (Fig. 7). A finite element model with 1577 nodes and 480 elements is constructed by an 8-node quadratic element (Fig. 8). The measurement temperature errors are set within −0.05152–0.05152, which implies that the average standard deviation of measurements is 0.02 for a 99% confidence bound (Fig. 9).

Three heat fluxes with the triangular time history are assumed to apply at the top surface of the slab, and they are estimated from the measured temperature at the bottom surface of the slab. When measurement errors are not included, that is, $\sigma = 0$ and $r = 1$, the results have good approximations (Fig. 10). With measurement errors, that is, $\sigma = 0.02$ and $r = 3$, included, the estimated results are still satisfied (Fig. 11). The results show that the maximum deviation of the inverse solution is about 0.02 when the measurement errors appear. The total number of iterations in this example is 1102 and 1007

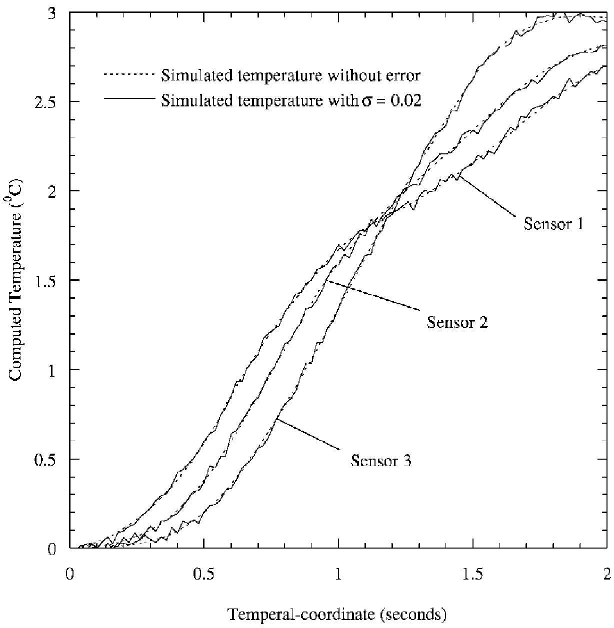

Fig. 9 Comparison of the calculated temperature from the multiflux pulse boundary conditions with measured temperatures in example 2 when $\sigma = 0.02$.

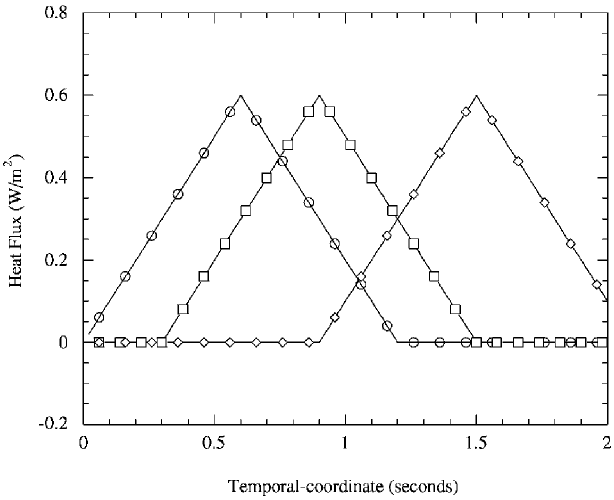

Fig. 10 Estimation of multiflux pulses in example 2 when $\sigma = 0$ and $r = 1$: ——, exact; ○, flux 1; □, flux 2; and ◇, flux 3.

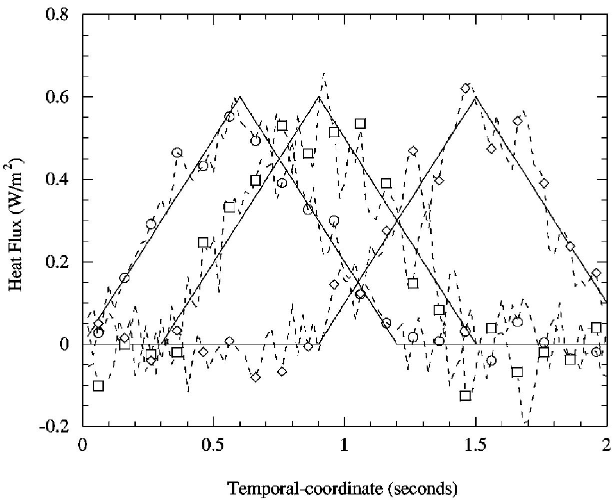

Fig. 11 Estimation of multiflux pulses in example 2 when $\sigma = 0.02$ and $r = 3$: ——, exact; ○, flux 1; □, flux 2; and ◇, flux 3.

when $\sigma = 0$ and $r = 1$ and $\sigma = 0.02$ and $r = 3$, respectively. In other words, the average number of iterations in each time step is about 11, which shows that the speed of convergence of the proposed method is fast.

From the preceding discussion, it can be concluded that the proposed method is an accurate, stable, and efficient method to determine the boundary conditions in inverse heat conduction problems.

Conclusions

A sequential method has been introduced to determine boundary conditions in nonlinear inverse conduction problems, that is, the inverse problem with the temperature-dependent thermal properties. The inverse solution at each time step is solved by a modified Newton–Raphson method. The method does not adopt the nonlinear least-squares error to formulate the inverse problem, but employes a direct comparison of the measured temperature and calculated temperature. Special features about this method are that no preselected functional form for the unknown function is necessary and that no nonlinear least squares is needed in the algorithm. Two examples have been given based on the proposed method. In the examples, the accuracy of the estimated results for the different types of boundary conditions are investigated. The results show that the heat flux estimation is more sensitive to the measurement error than the temperature estimation. The estimated results with different measurement errors are also discussed. Large errors are shown to make the estimated results diverge from the error-free solution. Furthermore, the average number of iterations in each time step is small, which indicates that the speed of convergence is fast in the proposed method. In conclusion, from the results in the examples, it appears that the proposed method is an accurate, stable, and efficient inverse technique. The proposed method is applicable to other kinds of nonlinear inverse problems, such as source strength estimation in the multidimensional inverse conduction problem.

References

[1] Tikhonov, A. N., and Arsenin, V. Y., *Solutions of Ill-Posed Problems*, 1st ed., Winston, Washington, DC, 1977.

[2] Beck, J. V., Blackwell, B., and St. Clair, C. R., *Inverse Heat Conduction-Ill Posed Problem*, 1st ed., Wiley, New York, 1985.

[3] Morozov, V. A., and Stessin, M., *Regularization Methods for Ill-Posed Problems*, 1st ed., CRC Press, Boca Raton, FL, 1993.

[4] Alifanov, O. M., *Inverse Heat Transfer Problems*, 1st ed., Springer-Verlag, Berlin, 1994.

[5] Liu, J., "A Stability Analysis on Beck's Procedure for Inverse Heat Conduction Problems," *Journal of Computational Physics*, Vol. 123, No. 1, 1995, pp. 65–73.

[6] Reinhardt, H. J., "A Numerical Method for the Solution of the Two-Dimensional Inverse Heat Conduction Problem," *International Journal for Numerical Methods in Engineering*, Vol. 32, No. 2, 1991, pp. 363–383.

[7] Alifanov, O. M., and Artyukhin, E. A., "Regularized Numerical Solution of Nonlinear Inverse Heat Conduction Problem," *Journal of Engineering Physics*, Vol. 29, No. 1, 1975, pp. 934–938.

[8] Alifanov, O. M., and Mikhailov, V. V., "Solution of the Nonlinear Inverse Thermal Conductivity Problem by the Iteration Method," *Journal of Engineering Physics*, Vol. 35, No. 6, 1978, pp. 1501–1506.

[9] Beck, J. V., and Murio, D. A., "Combined Function Specification–Regularization Procedure for Solution of Inverse Heat Conduction Problem," *AIAA Journal*, Vol. 24, No. 1, 1986, pp. 180–185.

[10] Busby, H. R., and Trujillo, D. M., "Numerical Solution to a Two-Dimensional Inverse Heat Conduction Problem," *International Journal for Numerical Methods in Engineering*, Vol. 21, 1985, pp. 349–359.

[11] Murio, D. A., *The Mollification Method and the Numerical Solution of Ill-Posed Problems*, Wiley-Interscience, New York, 1993.

[12] Jarny, Y., Ozisik, M. N., and Bardon, J. P., "A General Optimization Method Using Adjoint Equation for Solving Multidimensional Inverse Heat Conduction," *International Journal of Heat and Mass Transfer*, Vol. 34, No. 11, 1991, pp. 2911–2919.

[13] Li, H. Y., and Yang, C. Y., "A Genetic Algorithm for Inverse Radiation Problems," *International Journal of Heat and Mass Transfer*, Vol. 40, No. 7, 1997, pp. 1545–1549.

[14] Yang, C. Y., "A Sequential Method to Estimate the Strength of the Heat Source Based on Symbolic Computation," *International Journal of Heat and Mass Transfer*, Vol. 41, No. 14, 1998, pp. 2245–2252.

[15] Yang, C. Y., "Inverse Estimation of Mix-Typed Boundary Conditions in Heat Conduction Problems," *Journal of Thermophysics and Heat Transfer*, Vol. 12, No. 4, 1998, pp. 552–561.

[16] Beck, J. V., "Nonlinear Estimation Applied to the Nonlinear Inverse Heat Conduction Problem," *International Journal of Heat and Mass Transfer*, Vol. 13, 1970, pp. 703–716.

[17] Dorai, G. A., and Tortorelli, D. A., "Transient Inverse Heat Conduction Problem Solutions Via Newton's Method," *International Journal of Heat and Mass Transfer*, Vol. 40, No. 17, 1997, pp. 4115–4127.

[18] Yang, C. Y., "Estimation of the Temperature-Dependent Thermal Conductivity in Inverse Heat Conduction Problems," *Applied Mathematical Modelling*, Vol. 23, 1999, pp. 469–478.

[19] Hsu, T. R., Sun, N. S., Chen, G. G., and Gong, Z. L., "Finite Element Formulation for Two-Dimensional Inverse Heat Conduction Analysis," *Journal of Heat Transfer*, Vol. 114, Aug. 1992, pp. 553–557.

[20] Carnahan, B., Luther, H. A., and Wilkes, J. O., *Applied Numerical Methods*, 1st ed., Wiley, New York, 1977.

[21] Frank, N., and Wolfe, P., "An Algorithm for Quadratic Programming," *Naval Research Logistics Quarterly*, Vol. 3, 1956, pp. 95–110.

[22] "Math Library Version 1.0," User's Manual, IMSL Library Edition 10.0, International Mathematical and Statistical Libraries, Houston, TX, 1987.